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# Inherent irreversibility and thermal stability for steady flow of variable viscosity liquid film in a cylindrical pipe with convective cooling at the surface

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# Abstract

**Purpose** – The purpose of this paper is to investigate the inherent irreversibility and thermal stability in the flow of a variable viscosity fluid through a cylindrical pipe with convective cooling at the surface.

**Design/methodology/approach** – The non-linear momentum and energy equations governing the flow are solved analytically using a perturbation method coupled with a special type of Hermite-Padé approximation technique implemented numerically on MAPLE.

**Findings** – Expressions for dimensionless velocity and temperature, thermal criticality conditions and entropy generation number are obtained. A decrease in the fluid viscosity enhances both entropy generation rate and the dominant effect of heat transfer irreversibility near the wall

**Originality/value** – This paper presents the application of the second law of thermodynamics and a special type of Hermite-Padé approximation technique to variable viscosity cylindrical pipe flow with convective cooling at the wall.

**Keywords** Heat transfer, Viscosity, Liquid flow, Thermal stability, Convection, Coding **Paper type** Research paper

# 1. Introduction

Thuthuka programme for financial support.

The problem of flow and heat transfer is of considerable interest in many engineering and industrial applications such as metal extrusion, glass fiber and paper production, petroleum refinery, manufacturing of plastic and rubber sheets, polymer sheet extrusion from a dye, the drawing of plastic films, pipeline lubrication and processes in the chemical industry (Schlichting, 2000). However, it is well known that viscosity of most fluid encountered in engineering may change with temperature (Rigatos and Charalambakis, 2001). To accurately predict the flow and heat transfer rates, it is necessary to take into account this variation of viscosity. Viscosity variation with temperature in case of fluid clear of solid material was the subject of many studies and a complete literature survey may be found in Kays and Crawford (1993). These thermoviscous fluid are characterized by a strong dissipation, induced by the combined effect of heat diffusion and viscosity variation (Payr *et al.*, 2005). The nonlinear character of the material tends to destabilize the flow system, breaking the regular behavior of solutions of the corresponding differential equations (Wall and



International Journal of Numerical Methods for Heat & Fluid Flow Vol. 20 No. 1, 2010 pp. 5-16 © Emerald Group Publishing Limited 0961-5539 DOI 10.1108/09615531011008091

Variable viscosity liquid film

5

Received 20 August 2008 Revised 30 November 2008 Accepted 12 January 2009

The authors would like to thank the CSIR and National Research Foundation of South Africa

Wilson, 1996). Hence, it is very important to determine the thermal criticality conditions for such thermoviscous materials in order to maintain stability in the system (Makinde, 2007).

Moreover, it is now widely recognized that convective heat transfer problems that were previously studied using the first law of thermodynamics can be re-examined in the light of the second law of thermodynamics so that thermal systems can be designed with the objective of minimizing thermodynamic irreversibility (Bejan, 1979). This design methodology, known as entropy generation minimization, is comprehensively covered in the book by Bejan (1996). Different sources are responsible for generation of entropy such as heat transfer and viscous dissipation. The former is present in almost all of heat transfer devices due to heat transfer in finite temperature differences and the latter is responsible for dissipation of mechanical power to heat (viscous dissipation divided by the local absolute temperature) (Makinde, 2006; Tasnim and Mahmud, 2002; Sahin, 1999).

The objective of this study is to investigate the inherent irreversibility and thermal stability for steady flow of temperature-dependent viscosity fluids in a cylindrical pipe with convective cooling at the surface. The investigation is organized as follows: in sections 2 and 3, the physical problem is described, along with the governing equations and their non-dimensionalization. In section 4, we introduce and apply some rudiments of Hermite-Padé approximation technique (Wall and Wilson, 1996) in order to obtain the criticality conditions in the system. Section 5 describes the volumetric entropy generation rate, irreversibility distribution ratio and the Bejan number. The results are presented graphically and discussed quantitatively in section 6.

#### 2. Hydrodynamic and thermal analysis

Consider the steady flow of an incompressible variable viscosity liquid film through a circular cylindrical pipe with convective cooling at the surface (see Figure 1). It is assumed that the fluid motion is induced by applied axial pressure gradient and that the characteristic length in flow direction is typically large as compared with that across the film. This suggests that lubrication theory can be employed and the inertia terms in the governing momentum and energy balance equations can be easily neglected since we are dealing with a very small aspect ratio problem. Under these conditions the governing momentum and energy balance equations take the form (Schlichting, 2000).

$$\frac{1}{r}\frac{d}{dr}\left(\mu r\frac{du}{dr}\right) = -G, \\ \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \mu Br\left(\frac{du}{dr}\right)^2 = 0$$
(1)



Figure 1. Geometry of the problem

HFF

20.1

The appropriate boundary conditions in dimensionless form are given as follows: the pipe surface is fixed, impermeable and exchange heat with the ambient following Newton's cooling law:

$$u = 0, \quad \frac{dT}{dr} = -BiT \quad \text{at } r = 1$$
 (2)

and the axisymmetric condition along the pipe centerline, i.e.

$$\frac{du}{dr} = \frac{dT}{dr} = 0, \quad \text{at } r = 0 \tag{3}$$

where 0 < r < 1 is the dimensionless radius, *u* is the dimensionless velocity, *T* is the dimensionless temperature and the dimensionless viscosity  $\mu$  depends exponentially on temperature as (Kays and Crawford, 1993; Sahin, 1999):

$$\mu = e^{-\alpha\theta} \tag{4}$$

The above dimensionless governing equations (1)-(4) are obtained using the following variables and parameters:

$$r = \frac{\bar{r}}{H}, \quad z = \frac{\bar{z}}{L}, \quad u = \frac{\bar{u}}{U}, \quad \varepsilon = \frac{H}{L}, \quad \mu = \frac{\bar{\mu}}{\mu_0}, \quad T = \frac{\bar{T} - T_a}{T_i - T_a}, \quad G = -\frac{\partial P}{\partial z},$$

$$P = \frac{\varepsilon^2 L \bar{P}}{\mu_0 U}, \quad \alpha = \beta (T_i - T_a), \quad Br = \frac{\mu_0 U^2}{k (T_i - T_a)}, \quad Bi = \frac{hH}{k}.$$
(5)

where *H* is the radius, *h* is the heat transfer coefficient, *Br* is the Brinkman, *Bi* is the Biot number, *U* is the velocity scale,  $\mu_0$  is the fluid dynamic viscosity at the ambient temperature  $T_a$ , *P* the fluid pressure, *z* is the axial distance,  $T_i$  is the fluid initial temperature,  $\alpha$  the viscosity variation parameter, *k* is the thermal conductivity and *G* is the constant axial pressure gradient. Equations (1)-(4) can be easily combined to give

$$\frac{du}{dr} = -\frac{rG}{2}e^{\alpha T}, \quad \frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\lambda}{4}r^3e^{\alpha T} = 0 \tag{6}$$

where  $\lambda = BrG^2$ . In the following sections, equation (6) is solved using both perturbation and multivariate series summation techniques (Wall and Wilson, 1996).

# 3. Solution method

Due to the non-linear nature of the velocity and temperature field equations in (13), it is convenient to form a power series expansion both in the parameter  $\lambda$ , i.e.

$$u = \sum_{i=0}^{\infty} u_i \lambda^i, \quad T = \sum_{i=0}^{\infty} T_i \lambda^i$$
(7)

Substituting the solution series in equation (7) into equation (6) and collecting the coefficients of like powers of  $\lambda$ , we obtained and solved the equations for the coefficients of solution series iteratively. The solutions for the velocity and temperature

Variable viscosity liquid film

fields are given as

$$T(r) = -\frac{1}{64} \frac{(r^4 B i - 4 - B i)\lambda}{B i} + \frac{(1/16,394)\alpha(r^8 B i^2 - 16r^2 B i - 4r^4 B i^2)}{4r^4 B i^2} + O(\lambda^3)$$
(8)

1) /

$$u(r) = -\frac{1}{4}G(r-1)(r+1) + \frac{\times(Bir^4 + r^2Bi - 12 - 2Bi)\lambda}{Bi} - \frac{1}{9,83,040}G\alpha^2(r-1)$$
(9)  
(r+1)(9Bi^2r^8 + 9r^6Bi^2 - 160Bir^4 - 31r^4Bi^2 - 160r^2Bi - 31r^2Bi^2 + 44Bi^2 + 440Bi + 1440)\lambda^2/Bi^2 + O(\lambda)^2

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Using a computer symbolic algebra package (MAPLE, n.d.), the first few terms of the above solution series in equations (8) and (9) are obtained. We are aware that these power series solutions are valid for very small parameter values. However, using Hermite-Padé approximation technique, we have extended the usability of the solution series beyond small parameter values as illustrated in the following section.

#### 4. Thermal criticality study

The appearance of criticality or non-existence of steady-state solution to non-linear problems under investigation for certain parameter values is investigated using a multivariate series summation approach. Suppose that the partial sum

$$U_{N-1}(\lambda) = \sum_{i=0}^{N-1} a_i \lambda^i = U(\lambda) + O(\lambda^N) \text{ as } \lambda \to 0$$
(10)

is given (Bender and Orszag, 1978). It is important to note here that equation (10) can be used to approximate any output of the solution of the problem under investigation (e.g. the series for the wall heat flux parameter in terms of Nusselt number Nu = -dT/dr at r = 1), since everything can be Taylor expanded in the given small parameter. Assume  $U(\lambda)$  is a local representation of an algebraic function of  $\lambda$  in the context of non-linear problems, we seek an expression of the form

$$F_d(\lambda, U_{N-1}) = A_{0N}(\lambda) + A^d_{1N}(\lambda)U^{(1)} + A^d_{2N}(\lambda)U^{(2)} + A^d_{3N}(\lambda)U^{(3)}$$
(11)

such that

$$A_{0N}(\lambda) = 1, \ A_{iN}^{d}(\lambda) = \sum_{j=1}^{d+i} b_{ij} \lambda^{j-1}$$
(12)

and

$$F_d(\lambda, U) = O(\lambda^{N+1}) \text{ as } \lambda \to 0$$
 (13)

HFF 20.1

where  $d \ge 1, i = 1, 2, 3$ . Condition (12) normalizes the  $F_d$  and ensures that the order of series  $A_{iN}^{d}$  increases as *i* and *d* increase in value. There are thus 3(2 + d) undetermined coefficients  $b_{ij}$  in the expression (12). The requirement in equation (13) reduces the problem to a system of N linear equations for the unknown coefficients of  $F_d$ . The entries of the underlying matrix depend only on the N given coefficients  $a_i$  and we take N = 3(2 + d), so that the number of equations equals the number of unknowns. It is very important to note that the value of d depends on the N coefficients of the partial sum available. Equation (13) is a new special type of Hermite-Padé approximants. For instance, by letting  $U^{(1)} = U$ ,  $U^{(2)} = U^2$ ,  $U^{(3)} = U^3$ , we obtain a cubic Padé approximant. This enables us to determine the possible solution branches of the underlying problem in addition to the one represented by the original series. In the same manner, if we let  $U^{(1)} = U$ ,  $U^{(2)} = DU$ ,  $U^{(3)} = D^2U$ , in equation (13), where D is the differential operator given by  $D = d/d\lambda$ , we obtain a second-order differential approximants, which enables us to determine the dominant singularity or criticality in the flow field (i.e. by equating the leading polynomial coefficient  $A_{3N}(\lambda)$  of the equation (13) to zero). It is noteworthy that the rationale for chosen the degrees of  $A_{iN}$  in equation (11) for this particular application is based on the simple technique of singularity determination for second-order linear ordinary differential equations with polynomial coefficients as well as the possibility of multiple solution branches for the non-linear problem (Bender and Orszag, 1978). Following Guttamann (1989), it is well known that the dominant behavior of any output of the solution to a differential equation can be represented for some b and H as

$$U(\lambda) \approx \begin{cases} H(\lambda_C - \lambda)^b & \text{for } b \neq 0, 1, 2, \dots \\ H(\lambda_C - \lambda)^b \ln |\lambda_C - \lambda| & \text{for } b = 0, 1, 2, \dots \end{cases} \text{ as } \lambda \to \lambda_c$$
(14)

where *H* is a constant and  $\lambda_c$  is the critical point with the critical exponent *b*. Using Newton's polygon algorithm (Kays and Crawford, 1993), the critical exponent  $b_N$  can easily be determined. If we assume a singularity of algebraic type as in equation (14) with respect to our differential approximant in equation (13), then the critical exponent may be approximated by

$$b_N = 1 - \frac{A_{2N}(\lambda_{CN})}{DA_{3N}(\lambda_{CN})} \tag{15}$$

Generally, in the case of algebraic equations, the only singularities that are structurally stable are simple turning points. Hence, in practice, one almost invariably obtains  $b_N = 1/2$ .

### 5. Entropy generation

Flow and heat transfer processes inside narrow channel are irreversible. The nonequilibrium conditions arise due to the exchange of energy and momentum within the fluid and at solid boundaries, thus resulting in entropy generation. Apart of the entropy production is due to the heat transfer in the direction of finite temperature gradients and the other part of entropy production arises due to the fluid friction. The general equation for the entropy generation per unit volume is given by Tasnim and Mahmud (2002) Variable viscosity liquid film

$$S^m = \frac{k}{T_i^2} \left(\nabla \bar{T}\right)^2 + \frac{\mu}{T_i} \Phi \tag{16}$$

The first term in equation (16) is the irreversibility due to heat transfer and the second term is the entropy generation due to viscous dissipation. Using equation (16), we express the entropy generation number in dimensionless form as

$$Ns = \frac{H^2 T_i^2 S^m}{k(T_i - T_a)^2} = \left(\frac{\partial T}{\partial r}\right)^2 + \frac{\mu Br}{\Omega} \left(\frac{\partial u}{\partial r}\right)^2 \tag{17}$$

where  $\Omega = (T_i - T_a)/T_i$  is the temperature difference parameter. In equation (17), the first term can be assigned as  $N_1$  and the second term due to viscous dissipation as  $N_2$ , i.e.

$$N_1 = \left(\frac{\partial T}{\partial r}\right)^2, \quad N_2 = \frac{\mu B r}{\Omega} \left(\frac{\partial u}{\partial r}\right)^2$$
 (18)

In order to have an idea whether fluid friction dominates over heat transfer irreversibility or vice versa, Bejan (1979, 1996) defined the irreversibility distribution ratio as  $\Phi = N_2/N_1$ . Heat transfer dominates for  $0 \le \Phi < 1$  and fluid friction dominates when  $\Phi > 1$ . The contribution of both heat transfer and fluid friction to entropy generation are equal when  $\Phi = 1$ . In many engineering designs and energy optimization problems, the contribution of heat transfer entropy  $N_1$  to overall entropy generation rate  $N_s$  is needed. As an alternative to irreversibility parameter, the Bejan number (*Be*) is define mathematically as

$$Be = \frac{N_1}{Ns} = \frac{1}{1+\Phi} \tag{19}$$

Clearly, the Bejan number ranges from 0 to 1. Be = 0 is the limit where the irreversibility is dominated by fluid friction effects and Be = 1 corresponds to the limit where the irreversibility due to heat transfer by virtue of finite temperature differences dominates. The contribution of both heat transfer and fluid friction to entropy generation are equal when Be = 1/2.

# 6. Results and discussion

In order to numerically validate our results, physically meaningful values of the parameters entering into the problem are chosen. The axial pressure gradient parameter taken as G = 1, so that the viscous heating parameter  $\lambda$  is essentially equal to the Brinkmann number *Br*. The Hermite-Padé approximation procedure in section 4 above was applied to the first few terms of the solution series in section 3 and we obtained the results as shown in Tables I and II.

The rapid convergence of Hermite-Padé approximation procedure with gradual increase in the number of series coefficients utilized in the approximants is illustrated in Table I.

From Table II, it is interesting to note that the magnitude of thermal criticality for viscous heating parameter ( $\lambda_c$ ) increases with an increase in both convective cooling

10

HFF 20.1

and fluid viscosity (i.e.  $Bi \to \infty$  and  $\alpha \to 0$ ) hence, preventing the early development of thermal runaway and enhancing thermal stability.

In Figures 2-4, the velocity profiles are reported for increasing values of  $\lambda$ , Bi and  $\alpha$ . Generally, a parabolic velocity profile is observed with maximum value along the pipe centerline and minimum at the wall. The fluid velocity increases with increasing values of  $\lambda$  and  $\alpha$ , but decreases with increasing values of Bi.

Typical variations of the fluid temperature profiles in the normal direction are shown in Figures 5-7. The fluid temperature decreases with increasing values of Bi, increases with increasing values of  $\alpha$  and  $\lambda$ . Meanwhile, minimum temperature is generally observed at the wall due to convective cooling.

A slice of the bifurcation diagram for Bi > 0 in the  $(\lambda, Nu)$  plane is shown in Figure 8. It represents the variation of wall heat flux (Nu) with viscous heating parameter  $(\lambda)$ . In particular, for  $\alpha > 0$  there is a critical value  $\lambda_c$  (a turning point) such that, for  $0 \le \lambda < \lambda_c$  there are two solutions (labeled I and II). The upper and lower solution branches occur due to the temperature-dependent variable viscosity in the governing

d	N	Nu	$\lambda_{cN}$	$b_{cN}$	
				<u> </u>	Table I.
1	9	0.09876508294591	0.58130755126866	0.499879	Computations showing
2	12	0.09875019459897	0.58129568903072	0.499999	the procedure rapid
3	15	0.09875019525148	0.58129568935107	0.500000	convergence
4	18	0.09875019525148	0.58129568935107	0.500000	$(\alpha = 1, Bi = 0.1)$

Bi	$\alpha$	Nu	$\lambda_{cN}$	$b_{cN}$	
0.1	0.1	0.987501952514	5.812956893510	0.5000	
0.1	0.5	0.197500390502	1.162591378702	0.5000	
0.1	1.0	0.098750195251	0.581295689351	0.5000	Table II.
1.0	1.0	0.876894374382	5.197760638083	0.5000	Computations showing
10.0	1.0	3.2296703857309	22.308675312695	0.5000	thermal criticality
100.0	1.0	3.9200319744255	30.757560158350	0.5000	for different
$\infty$	1.0	4.00000000000000	32.000000000000	0.5000	parameter values



Figure 2. Velocity profile: Bi = 1;  $\alpha = 2$ ;  $\lambda = 0.1$ ;  $00000 \lambda = 0.5$ ;  $++++\lambda = 1.0$ 

Variable viscosity liquid film

thermal boundary layer equation (equation (6)). When  $\lambda_c < \lambda$ , the system has no real solution and displays a classical form indicating thermal runaway.

HFF

20,1

12

Figures 9 and 10 illustrate the entropy generation rate in the transverse direction for various parametric values. It is noteworthy that entropy generation rate is at the lowest in the region around the pipe centerline and increases quite rapidly near the wall with maximum value at the wall. We observe that a decrease in the fluid viscosity results into a further increase in the entropy generation rate at the wall, whereas an increase in *Bi* results into a decrease entropy generation rate at the wall.



In Figures 11-13, the Bejan (*Be*) number is illustrated for various parametric values. It is observed that the fluid friction irreversibility dominates around the pipe centreline region, whereas near the wall heat transfer irreversibility dominates. The dominant effect of heat transfer irreversibility near the wall further increases with increasing

Variable viscosity liquid film



values of  $\alpha$  and group parameter ( $Br\Omega^{-1}$ ) but decreases with increasing effect of HFF convective cooling at the pipe surface.

# 7. Conclusion

This paper presents the application of the second law of thermodynamics and a special type of Hermite-Padé approximation technique to temperature-dependent viscosity cylindrical pipe flow with convective cooling at the wall. The velocity and temperature profiles are obtained and used to evaluate the entropy generation number. Our bifurcation procedure reveals accurately the thermal criticality conditions and the solution branches. For all parametric values, viscous dissipation irreversibility dominates around the pipe centerline, whereas near the wall the heat transfer



14

20.1

irreversibility dominates. A decrease in the fluid viscosity enhances both entropy generation rate and the dominant effect of heat transfer irreversibility near the wall. In the future, this work can be modified in a number of directions such as: thermal stability and second-law analysis in concentric pipe flow; effect of other models of temperature-dependent viscosity. Variable viscosity liquid film



#### References

- Bejan, A. (1979), "A study of entropy generation in fundamental convective heat transfer", ASME Journal of Heat Transfer, Vol. 101, pp. 718-25.
- Bejan, A. (1996), Entropy Generation Minimization, CRC Press, Boca Raton, FL.
- Bender, C. and Orszag, S.A. (1978), Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, New York, NY.
- Guttamann, A.J. (1989), "Asymptotic analysis of power-series expansions", in Domb, C. and Lebowitz, J.K. (Eds), *Phase Transitions and Critical Phenomena*, Academic Press, New York, NY, pp. 1-234.
- Kays, W.M. and Crawford, M.E. (1993), *Convective Heat and Mass Transfer*, McGraw-Hill, New York, NY.
- Makinde, O.D. (2006), "Irreversibility analysis for gravity driven non-Newtonian liquid film along an inclined isothermal plate", *Physica Scripta*, Vol. 74, pp. 642-5.
- Makinde, O.D. (2007), "Hermite-Padé approximation approach to steady flow of a liquid film with adiabatic free surface along an inclined heat plate", *Physica A*, Vol. 381, pp. 1-7.
- MAPLE (n.d.), available at: http://maplesoft.com/products/maple/technical.aspx

HFF 20,1	<ul> <li>Payr, M., Vanaparthy, S.H. and Meiburg, E. (2005), "Influence of variable viscosity on density-driven instabilities in capillary tubes", <i>Journal of Fluid Mechanics</i>, Vol. 525, pp. 333-53.</li> <li>Rigatos, A.P. and Charalambakis, N.C. (2001), "Two-dimensional adiabatic Newtonian flow with temperature dependent viscosity", <i>International Journal of Engineering Science</i>, Vol. 39, pp. 1143-65.</li> </ul>
16	Sahin, A.Z. (1999), "Effect of variable viscosity on the entropy generation and pumping power in a laminar fluid flow through a duct subjected to constant heat flux", <i>Heat and Mass Transfer</i> , Vol. 35, pp. 499-506.
	Schlichting, H. (2000), Boundary Layer Theory, Springer-Verlag, New York, NY.
	Tasnim, S.H. and Mahmud, S. (2002), "Entropy generation in a vertical concentric channel with temperature dependent viscosity", <i>International Communication in Heat and Mass Transfer</i> , Vol. 29 No. 7, pp. 907-18.
	Wall, D.P. and Wilson, S.K. (1996), "The linear stability of channel flows of fluid with temperature dependent viscosity", <i>Journal of Fluid Mechanics</i> , Vol. 323, pp. 107-32.

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